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ON THE PROPAGATION OF PLANE SOUND WAVES IN DUCTS CARRYING AN INCOMPRESSIBLE AXIAL MEAN FLOW HAVING AN ARBITRARY VELOCITY PROFILE

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1. INTRODUCTION

In applications where it is necessary to study the transmission of plane sound waves in subsonic, low Mach number flow ducts at low frequencies, it is usually satisfactory to assume the mean flow to have a uniform velocity profile over the duct cross-section, although the actual profile may vary from a parabolic shape, characterizing a laminar flow, to a flat shape, characterizing a fully developed turbulent flow. This is based on previous solutions in 3-D on isentropic sound propagation in ducts with a parallel mean shear flow (e.g., references [1–5]; the literature dealing with the problem is quite extensive), which show that the fundamental mode propagation is only slightly dispersive and the uniform profile approximation gives a fairly good representation of the propagation constants and the cut-off frequency. However, a plane wave analysis of the problem is not seen, and it is the purpose of the present paper to provide this formulation. It transpires from this analysis that, for plane sound waves in uniform ducts, an arbitrary mean flow velocity profile is formally equivalent to a uniform mean flow and, therefore, the practical transmission calculations developed for the case of uniform mean flow can easily be refined for the effect of the mean flow velocity profile.

2. ANALYSIS

Consider a straight hard-walled duct that carries a steady axial mean flow. The propagation of plane sound waves in this fluid flow is governed by the linearized continuity and momentum equations: the continuity equation is

$$S\frac{\partial}{\partial t}(\rho_0+\rho)+\frac{\partial}{\partial x}\left[(\rho_0+\rho)\int_S(v_0+v)\,\mathrm{d}S\right]=0 \tag{1}$$

and the momentum equation is

$$\frac{\partial}{\partial t}\left[(\rho_0+\rho)\int_S(v_0+v)\,\mathrm{d}S\right]+\frac{\partial}{\partial x}\left[(\rho_0+\rho)\int_S(v_0+v)^2\,\mathrm{d}S\right]+S\frac{\partial}{\partial x}(p_0+p)=0. \tag{2}$$

Here, S denotes the cross-sectional area of the duct, x denotes the co-ordinate along the duct axis and t denotes the time. The fluid density and pressure and the particle velocity in

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the axial direction are assumed to consist of acoustic fluctuations ρ, p and v superimposed on the time-averaged mean values ρ_0, p_0 and v_0 respectively. The acoustic fluctuations are assumed to be functions of t and x only; v_0 is assumed to be independent of time, and p_0 is assumed to be constant. Neglecting the visco-thermal effects allows the density and pressure perturbations to be isentropically related, so that $\mathrm{d}p = c_0^2\,\mathrm{d}\rho$, where c_0 denotes the speed of sound. Thus, the first order equations governing the propagation of plane sound waves can now be obtained as usual, by expanding equations (1) and (2), neglecting products of acoustic perturbations as second order small quantities. Here, this process is applied by assuming the mean flow to be incompressible, which implies that the products $S\bar{v}_0$ and $S\beta\bar{v}_0^2$ are constant, where β and $\bar{v}_0$ are defined by equations (5) and (6). Hence, the acoustic continuity and momentum equations are obtained as, respectively,

$$\frac{\partial p}{\partial t} + \bar{v}_0\frac{\partial p}{\partial x} + \rho_0 c_0^2\left(\frac{\partial v}{\partial x} + \frac{\mathrm{d}S}{S\,\mathrm{d}x}\,v\right) = 0, \qquad \rho_0\left(\frac{\partial v}{\partial t} + \bar{v}_0\frac{\partial v}{\partial x} + \frac{\mathrm{d}\bar{v}_0}{\mathrm{d}x}\,v\right) + \alpha^2\frac{\partial p}{\partial x} = 0. \quad (3, 4)$$

Here,

$$\alpha^2 = 1 + (\beta - 1)\bar{M}_0^2, \qquad \beta = \frac{\int_S v_0^2\,\mathrm{d}S}{S\bar{v}_0^2}, \quad \bar{M}_0 = \frac{\bar{v}_0}{c_0}, \quad (5)$$

where $\bar{v}_0$ denotes the average mean flow velocity over the duct cross-sectional area,

$$\bar{v}_0 = \frac{1}{S}\int_S v_0\,\mathrm{d}S. \quad (6)$$

For non-uniform ducts, or for a uniform duct carrying an axially non-uniform flow (v_0 is a function of x also), equations (3) and (4) do not admit an obvious analytical solution and have to be solved numerically; however, for a uniform duct carrying an axially uniform mean flow, equations (3) and (4) simplify to, respectively,

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \bar{M}_0\frac{\partial}{\partial x}\right)p + z_0\frac{\partial v}{\partial x} = 0, \qquad \alpha^2\frac{\partial p}{\partial x} + z_0\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \bar{M}_0\frac{\partial}{\partial x}\right)v = 0, \quad (7, 8)$$

where z_0 ($=\rho_0 c_0$) denotes the characteristic impedance. Now, since α is a constant, equations (7) and (8) can be solved analytically for the propagation constants, but it is not necessary to undertake this step because, defining an effective characteristic impedance as $z_e = \rho_0 c_0/\alpha$, and an effective speed of sound as $c_e = c_0\alpha$, reduces these equations to the uniform mean flow profile ($\alpha = 1$) form:

$$\left(\frac{1}{c_e}\frac{\partial}{\partial t} + M_e\frac{\partial}{\partial x}\right)p + z_e\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial p}{\partial x} + z_e\left(\frac{1}{c_e}\frac{\partial}{\partial t} + M_e\frac{\partial}{\partial x}\right)v = 0, \quad (9, 10)$$

where $M_e = \bar{v}_0/c_e$ (the density fluctuations are still determined by the relationship $\mathrm{d}p = c_0^2\,\mathrm{d}\rho$). The solution of equations (9) and (10) is well known: assuming $\exp(-\mathrm{i}\omega t)$ time dependence, where (i denotes the unit imaginary number and ω is the radian frequency, it can be expressed as [6]

$$p(x) = p^+(x) + p^-(x), \quad (11)$$

$$v(x) = \frac{p^+(x) - p^-(x)}{z_e} = \alpha\,\frac{p^+(x) - p^-(x)}{\rho_0 c_0}, \quad (12)$$

where the pressure wave components are given by

$$p^{\mp}(x) = p^{\mp}(0)\exp(\mathrm{i}kK^{\mp}x). \quad (13)$$

Here, $k = \omega/c_0$ denotes the wavenumber, and

$$kK^{\mp} = \frac{\mp k_e}{1 \mp M_e}, \tag{14}$$

where $k_e = \omega/c_e$. Therefore, the propagation constants, $K^{\mp}$, are given by

$$K^{\mp} = \frac{\mp 1}{\alpha \mp \bar{M}_0}. \tag{15}$$

This result shows that the correction due to the mean flow velocity profile in the plane wave propagation constants is of $O[\bar{M}_0^2]$ and, therefore, its effects would be discernible for moderately high subsonic Mach numbers. Also, the closer β is to unity, the smaller is the effect of profile on the propagation constants, even if the mean flow Mach number is moderately high.

For laminar flow, for example, $\bar{M}_0 = M_c/2$ and $\beta = 4/3$, where M_c denotes the centreline Mach number. An exact analytical solution for this case has been presented in reference [5] assuming $(KM_0)^2 \ll 1$, where K denotes a propagation constant, and solutions were given for the propagations constants for $M_c = 0{\cdot}05$, 0·1 and 0·2. Equation (15) predicts the results of reference [5], to the accuracy attainable from the graphical data presented, with an error less than 1%.

In reference [4], the fundamental mode propagation constants have been plotted as functions of frequency for $M_c = 0{\cdot}2$ and 0·6, but no direct comparison with these values could be made here because the corresponding $\bar{M}_0$ and β could not be computed from the data provided. However, an estimation can be made by assuming an equivalent "1/n"th power-law turbulent flow that has the same profile-averaged Mach number. The latter is given in reference [4] as $M_e = 0{\cdot}18168$ for $M_c = 0{\cdot}2$, which corresponds to about "1/9·917" power-law. Hence, the propagation constants are found from equation (15) as $K^+ = 0{\cdot}8524$ and $K^- = -1{\cdot}209$. From the table presented for the forward wave for $M_c = 0{\cdot}2$, the corresponding value of reference [4] is $K^+ = 0{\cdot}8533$, which confirms the accuracy of the present prediction. Similarly, $M_c = 0{\cdot}6$ ($M_e = 0{\cdot}55$) corresponds to "1/11" power law, given the propagation constants as 0·6550 and $-$ 2·104. In this case, the corresponding results of reference [4] are not tabulated and their extraction as slopes of the supplied dispersion curves with definite accuracy is difficult; however, the above values appear to be very close to the predictions of reference [4].

A somewhat far-fetched application of this theory is a duct carrying a confined core flow. Assuming an axially uniform core flow, plane sound wave propagation in such a duct can be modelled by the present theory by taking $\beta = S/S_j$ and $\bar{M}_0 = M/\beta$, where S and S_j denote the cross-sectional area of the duct and the core, respectively, and M denotes the Mach number of the core flow. With this notation, the propagation constants for the forward and backward waves in the duct can be expressed as

$$K^{\pm} = \pm[\sqrt{1 + (\beta - 1)M^2/\beta^2} \pm M/\beta]^{-1}. \tag{16}$$

An exact dispersion equation for sound propagation in a circular duct carrying a confined axially uniform core flow has been derived in reference [7]. In Figure 1, the forward wave propagation constants as computed by using equation (16) and the dispersion equation derived in reference [7], for $\beta = 2$, are compared. Also shown in Figure 1 is the propagation constant for the core flow averaged over the duct cross-section as uniform mean flow; that is, $K^+ = 1/(1 + M/\beta)$. Equation (16) overestimates the predictions of reference [7] by less than about 5% error, but it provides about as much improvement over the values based on the assumption of uniform mean flow.

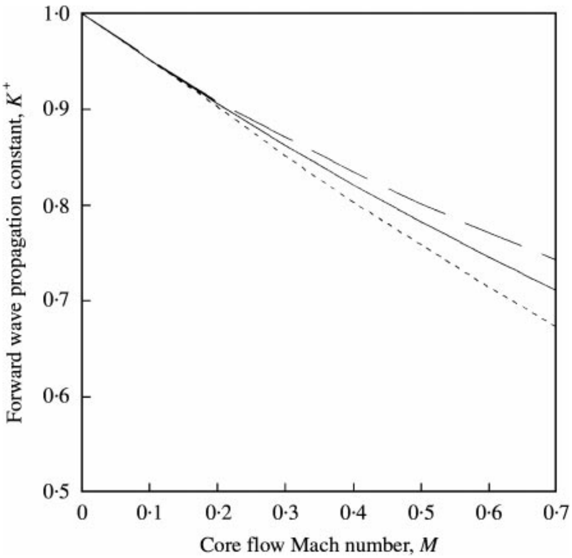

Figure 1. Propagation constant of the forward travelling wave in a duct carrying an axially uniform core flow, $\beta = 2$. ————, Present plane wave theory; - - - -, reference [7]; — — —, uniform profile with velocity averaged over duct cross-sectional area.

3. CONCLUSION

The theory of plane sound wave propagation in a uniform duct carrying a uniform mean flow has been extended to include the effect of the velocity profile. This theory may be used to refine practical duct acoustics calculations when frequency and mean flow velocity Mach number are low enough for diffraction and radial dependence due to shear flow is negligible.

REFERENCES

1. P. E. Doak and P. G. Vaidya 1970 *Journal of Sound and Vibration* **12**, 201–224. Attenuation of plane wave and higher order mode sound propagation in lined ducts.
2. W. Eversman 1972 *Journal of Sound and Vibration* **24**, 459–469. Representation of 1/Nth power law boundary layer in the sheared flow acoustic transmission problem.
3. P. N. Shankar 1972 *Journal of Sound and Vibration* **22**, 221–232. Sound propagation in duct shear layers.
4. N. K. Agarwal and M. K. Bull 1989 *Journal of Sound and Vibration* **132**, 275–298. Acoustic wave propagation in a pipe with fully developed turbulent flow.
5. G. R. Gogate and M. L. Munjal 1993 *Journal of Sound and Vibration* **160**, 465–484. Analytical solution of sound propagation in lined or unlined circular ducts with laminar mean flow.
6. P. O. A. L. Davies 1988 *Journal of Sound and Vibration* **124**, 91–113. Practical flow duct acoustics.
7. A. Cummings 1975 *Journal of Sound and Vibration* **38**, 149–155. Sound transmission at sudden area expansions in circular ducts with superimposed mean flow.